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THE STRONG M_α-INTEGRAL OF BANACH-VALUED FUNCTIONS

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Abstract. In this paper, we define the strong M_α-integral of Banach-valued functions and investigate some properties of the strong M_α-integral.

1. Introduction and preliminaries

It is well-known [12] that a function $f : [a,b] \to X$ is M_α-integrable on $[a,b]$ if and only if there exists an ACG_α function F such that $F' = f$ almost everywhere on $[a,b]$.

In this paper, we define the strong M_α-integral of Banach-valued functions and prove that a Banach-valued function f is strongly M_α-integrable on $[a,b]$ if and only if there exists a strong AC_α function F such that $F' = f$ almost everywhere on $[a,b]$.

Throughout this paper, $I_0 = [a,b]$ is a compact interval in R and X is a Banach space. Let D be a finte collection of interval-point pairs $\{(\xi_i, I_i)\}_{i=1}^n$, where $\{I_i\}_{i=1}^n$ are non-overlapping subintervals of I_0, and let δ be a positive function on I_0, i.e. $\delta : I_0 \to R^+$. We say that $D = \{(\xi_i, I_i)\}_{i=1}^n$ is

(1) a partial tagged partition of I_0 if $\cup_{i=1}^n I_i \subset I_0$,

(2) a tagged partition of I_0 if $\cup_{i=1}^n I_i = I_0$,

(3) a δ-fine McShane partition of I_0 if $I_i \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ and $\xi_i \in I_i$ for all $i = 1, 2, ..., n$,

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(4) a δ-fine M_α-partition of I_0 for a constant $\alpha > 0$ if it is a δ-fine McShane partition of I_0 and satisfying the

$$\sum_{i=1}^{n} dist(I_i, \xi_i) < \alpha,$$

where $dist(I_i, \xi_i) = inf\{|t - \xi_i| : t \in \xi_i\}$,

(5) a δ-fine Henstock partition of I_0 if $\xi_i \in I_i \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i)$ for all $i = 1, 2, ..., n$.

2. The strong M_α-integral

The strong Henstock integral and strong McShane integral are defined in [13].

Definition 2.1. [14] A function $f : I_0 \to X$ is strongly Henstock integrable on I_0 if there is an additive function F on I_0 such that for every $\epsilon > 0$ there exists a gauge δ on I_0 such that

$$\sum_{i=1}^{n} ||f(t_i)\mu(I_i) - F(I_i)|| < \epsilon$$

for every $\delta-$fine Henstock partition $\{(t_i, I_i), i = 1, 2, ..., n\}$ of I_0.

Definition 2.2. [14] A function $f : I_0 \to X$ is strongly McShane integrable on I_0 if there is an additive function F on I_0 such that for every $\epsilon > 0$ there exists a gauge δ on I_0 such that

$$\sum_{i=1}^{n} ||f(t_i)\mu(I_i) - F(I_i)|| < \epsilon$$

for every $\delta-$fine McShane partition $\{(t_i, I_i), i = 1, 2, ..., n\}$ of I_0.

Now we define the strong M_α-integral.

Definition 2.3. A function $f : I_0 \to X$ is strongly M_α-integrable on I_0 if there is an additive function F on I_0 such that for every $\epsilon > 0$ there exists a gauge δ on I_0 such that

$$\sum_{i=1}^{n} ||f(t_i)\mu(I_i) - F(I_i)|| < \epsilon$$

for every $\delta-$fine M_α-partition $\{(t_i, I_i), i = 1, 2, ..., n\}$ of I_0.

Since every Henstock partition is an M_α-partition and every M_α-partition is a McShane partition, we can easily get the following theorem.

THEOREM 2.4. *Let $f : I_0 \to X$ be a function.*
(a) If f is strongly McShane integrable on I_0, then f is strongly M_α-integrable on I_0.
(b) If f is strongly M_α-integrable on I_0, then f is strongly Henstock-integrable on I_0.

THEOREM 2.5. *If $f : I_0 \to X$ is strongly M_α-integrable on I_0 with the primitive F, then f is M_α-integrable on I_0 and $(M_\alpha)\int_{I_0} f = F(I_0)$.*

Proof. The result follows easily fron the inequality

$$\begin{aligned}\Big|\Big|\sum_{i=1}^{n} f(t_i)\mu(I_i) - F(I_0)\Big|\Big| &= \Big|\Big|\sum_{i=1}^{n}[f(t_i)\mu(I_i) - F(I_i)]\Big|\Big| \\ &\leq \sum_{i=1}^{n}\Big|\Big|f(t_i)\mu(I_i) - F(I_i)\Big|\Big|\end{aligned}$$

for every M_α-partition $\{(t_i, I_i), i = 1, 2, ..., n\}$ of I_0. □

DEFINITION 2.6. *A function $F : I_0 \to X$ is strongly differentiable at $c \in I_0$ if there is a $w \in X$ such that for every $\epsilon > 0$ there exists $\eta > 0$ such that*

$$\Big|\Big|\frac{F(y) - F(x)}{y - x} - w\Big|\Big| < \epsilon$$

for every interval $[x, y] \subset (c - \eta, c + \eta) \cap I_0$. We denote $w = F'_{st}(c)$ the strong derivative of F at c.

THEOREM 2.7. *If F is strongly differentiable on I_0, then F'_{st} is strongly M_α-integrable on I_0 and $\int_{I_0} F'_{st} = F(I_0)$.*

Proof. Let $\epsilon > 0$. For each $x \in I_0$, use the strong differentiability at x to choose $\delta(x) > 0$ so that $u, v \in I_0 \cap \big(x - \delta(x), x + \delta(x)\big)$ implies $||F(v) - F(u) - F'_{st}(x)(v - u)|| < \epsilon|v - u|$.

Suppose that $D = \big(x_i, [u_i, v_i]\big)$ is a δ-fine M_α-partition of I_0. Then $\sum_i \big|\big|F'_{st}(x_i)(v_i - u_i) - [F(v_i) - F(u_i)]\big|\big| < \sum_i \epsilon(v_i - u_i) = \epsilon\mu(I_0)$.

Hence, F'_{st} is strongly M_α-integrable on I_0 and

$$\int_{I_0} F'_{st} = F(I_0).$$

□

Definition 2.8. *Let $F : I_0 \to X$ and let E be a subset of I_0. Then F is strongly AC_α on E if for each $\epsilon > 0$ there is a constant $\eta > 0$ and a gauge $\delta : I_0 \to R^+$ such that*

$$\sum_i \big|\big|F(I_i)\big|\big| < \epsilon$$

for every δ-fine M_α-partial partition $D = \{(\xi_i, I_i)\}$ of I_0 satisfying $\xi_i \in E$ and $\sum_i \mu(I_i) < \eta$.

We can show that a function $f : I_0 \to X$ is strongly M_α-integrable on I_0 if and only if there exists a strong AC_α function F such that $F' = f$ almost everywhere on I_0.

Theorem 2.9. *If a function $f : I_0 \to X$ is strongly M_α-integrable on I_0 with the primitive F, then F is strongly AC_α on I_0 and $F' = f$ almost everywhere on I_0.*

Proof. Suppose that f is strongly M_α-integrable on I_0 with the primitive F. Since f is strongly Henstock integrable on I_0, $F'(t) = f(t)$ almost everywhere by [14, Theorem 7.4.2].

To show that F is strongly AC_α on I_0, let $\epsilon > 0$. Since f is strongly M_α-integrable on I_0, there exists a gauge δ on I_0 such that

$$\sum_{i=1}^{q} \Big|\Big|f(t_i)(v_i - u_i) - [F(v_i) - F(u_i)]\Big|\Big| < \epsilon$$

for every δ-fine M_α-partition $\{(t_i, [u_i, v_i]), i = 1, 2, ..., q\}$ of I_0. By [14, Theorem 7.4.3], f is continuous on I_0. Hence, there exists a real number $K > 0$ such that $||f(t)|| \leq K$ for all $t \in I_0$. Let $\eta = \frac{\epsilon}{K+1}$. Suppoe that $D = \{(\xi_i, [\alpha_j, \beta_j]), j = 1, 2, ..., p\}$ is a δ-fine M_α-partition of I_0 with $\sum_{j=1}^{p}(\beta_j - \alpha_j) < \eta$. Let $\beta = \alpha - \sum_{j=1}^{p} dist\big(\xi_i, [\alpha_j, \beta_j]\big)$. Then the closure $\overline{I_0 - \cup_{j=1}^{p}[\alpha_j, \beta_j]}$ cosists of a finite number of disjoint closed subintervals $I_k \subset I_0$ $(1 \leq k \leq n)$.

Taking any δ-fine $M_{\frac{\beta}{n}}$-partition D_k of I_k, $D \cup [\cup_{k=1}^{n} D_k] = \{(\tau_l, [c_l, d_l]), l = 1, 2, ..., p+n\}$ is a δ-fine M_α-partition of I_0. Then

$$\begin{aligned}
&\sum_{j=1}^{p} \big|\big|F(\beta_j) - F(\alpha_j) - f(\xi_j)(\beta_j - \alpha_j)\big|\big| \\
&\leq \sum_{l=1}^{p+n} \big|\big|F(d_l) - F(c_l) - f(\tau_l)(d_l - c_l)\big|\big| < \epsilon
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{j=1}^{p} \big|\big|F(\beta_j) - F(\alpha_j)\big|\big| \\
&\leq \sum_{j=1}^{p} \big|\big|F(\beta_j) - F(\alpha_j) - f(\xi_j)(\beta_j - \alpha_j)\big|\big| + \sum_{j=1}^{p} ||f(\xi_j)||(\beta_j - \alpha_j) \\
&\leq \epsilon + K\sum_{j=1}^{p}(\beta_j - \alpha_j) \\
&< \epsilon + K\eta \\
&< \epsilon + \epsilon = 2\epsilon.
\end{aligned}$$

Hence, F is strongly AC_α on I_0. □

THEOREM 2.10. *Let $F : I_0 \to X$ is strongly AC_α on I_0 and $F' = f$ almost everywhere on I_0, then f is strongly M_α-integrable on I_0.*

Proof. Suppose that F is strongly AC_α on I_0 and $F' = f$ almost everywhere on I_0. Let E be the set of points t at which $F'(t) \neq f(t)$. Then $\mu(E) = 0$.

For each $t \in I_0 - E$, given $\epsilon > 0$ there is a $\delta(t) > 0$ such that whenever $t \in [u, v] \subset \big(t - \delta(t), t + \delta(t)\big)$ we have

$$||F([u, v]) - f(t)(v - u)|| < \epsilon(v - u).$$

For each $j \in N$, let $E_j = \{t \in E : j - 1 \leq ||f(t)|| < j\}$. Then $\{E_j\}$ is a collection of pairwise disjoint measurable sets and $\cup_{j=1}^{\infty} E_j = E$. Since F is strongly AC_α on I_0, F is also strongly AC_α on E_j. Hence, there is a $\eta_j < \frac{\epsilon 2^{-j}}{j}$ such that for any δ-fine M_α-partial partition $\{(t_k, I_k)\}$ of I_0 satisfying $t_k \in E_j$ and $\sum_k \mu(I_k) < \eta_j$, we have

$$\sum_k ||F(I_k)|| < \epsilon 2^{-j}.$$

For each $j \in N$, choose an open set G_j such that $\mu(G_j) < \eta_j$ and $E_j \subset G_j$.

Now for $t \in E_j$ $(j = 1, 2, ...)$, put $\delta(t) > 0$ such that $(t - \delta(t), t + \delta(t)) \subset G_j$. Taking any δ-fine M_α-partition $\{t_l, [u_l, v_l])\}$ of I_0, we have

$$
\begin{aligned}
&\sum_l \big|\big|f(t_l)(v_l - u_l) - F([u_l, v_l])\big|\big| \\
&= \sum_{t_l \notin E} \big|\big|f(t_l)(v_l - u_l) - F([u_l, v_l])\big|\big| + \sum_{t_l \in E} \big|\big|f(t_l)(v_l - u_l) - F([u_l, v_l])\big|\big| \\
&\leq \sum_j \sum_{t_l \notin E_j} \big|\big|f(t_l)(v_l - u_l) - F([u_l, v_l])\big|\big| + \sum_j \sum_{t_l \in E_j} ||f(t_l)||(v_l - u_l) \\
&\qquad\qquad\qquad + \sum_j \sum_{t_l \in E_j} \big|\big|F([u_l, v_l])\big|\big| \\
&< \epsilon\mu(I_0) + \sum_j j\eta_j + \sum_j \epsilon 2^{-j} \\
&< \epsilon\mu(I_0) + 2\epsilon.
\end{aligned}
$$

This shows that f is strongly M_α-integrable on I_0 and $\int_{I_0} f = F(I_0)$. $\square$

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